Partial Dual and Delta-Matroid

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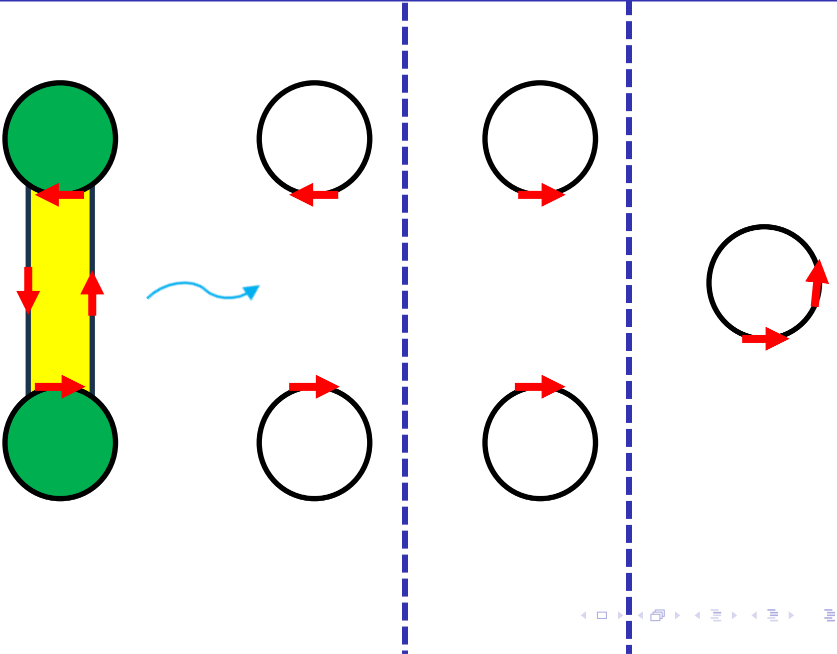
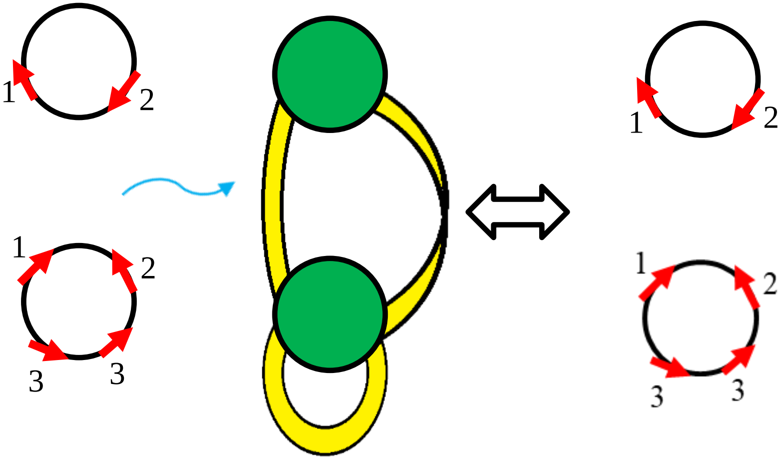
June 26, 2026

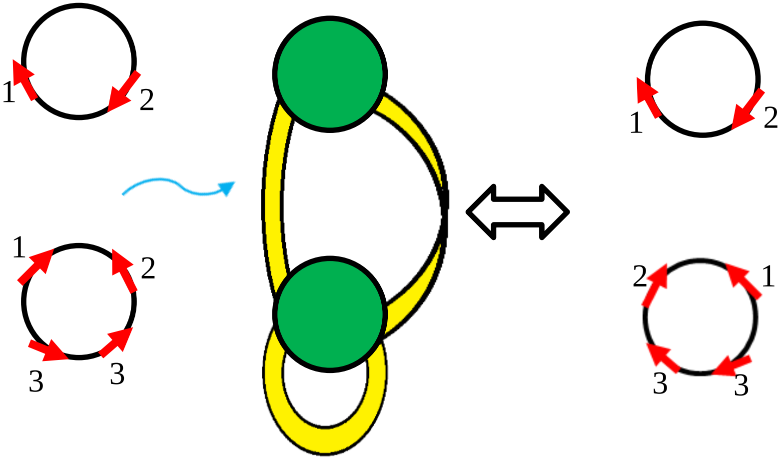
A combinatorial definition of ribbon graphs



A combinatorial definition of ribbon graphs



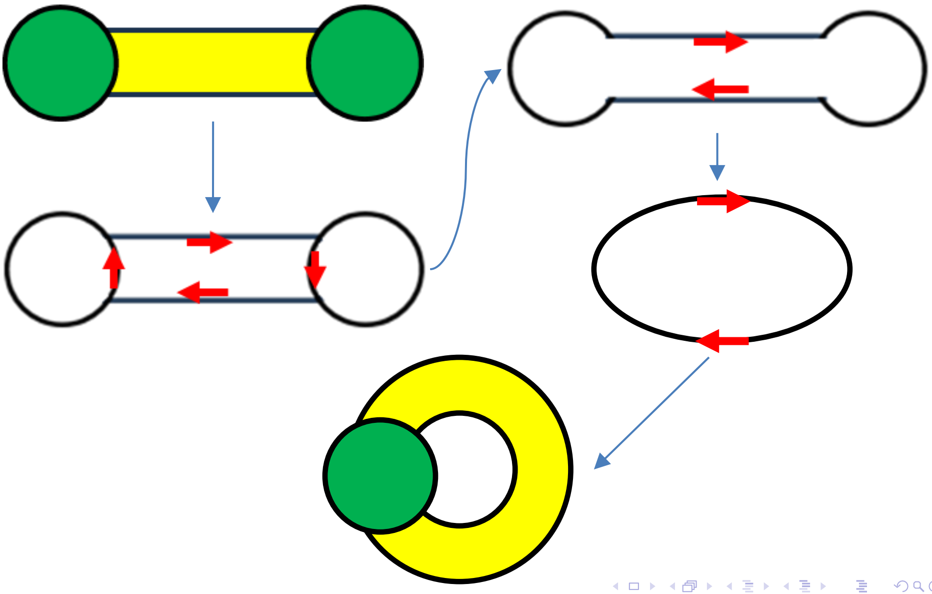
A combinatorial definition of ribbon graphs



Some notations

Notation.
Let $G = (V, E)$ be a ribbon graph, $A \subseteq E$, and e $\in E$. Then
(1) $G^{\delta(A)}$ means partial dual on A;
(2) $G^{\delta(e)}$ means partial dual on e;
(3) $G^* := G^{\delta(E)}$;
(4) $G^A := G^{\delta(A)}$ in some paper.

The table of Partial Dual



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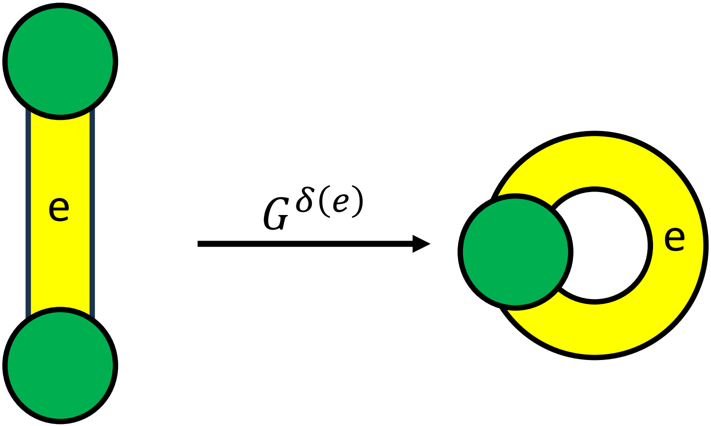

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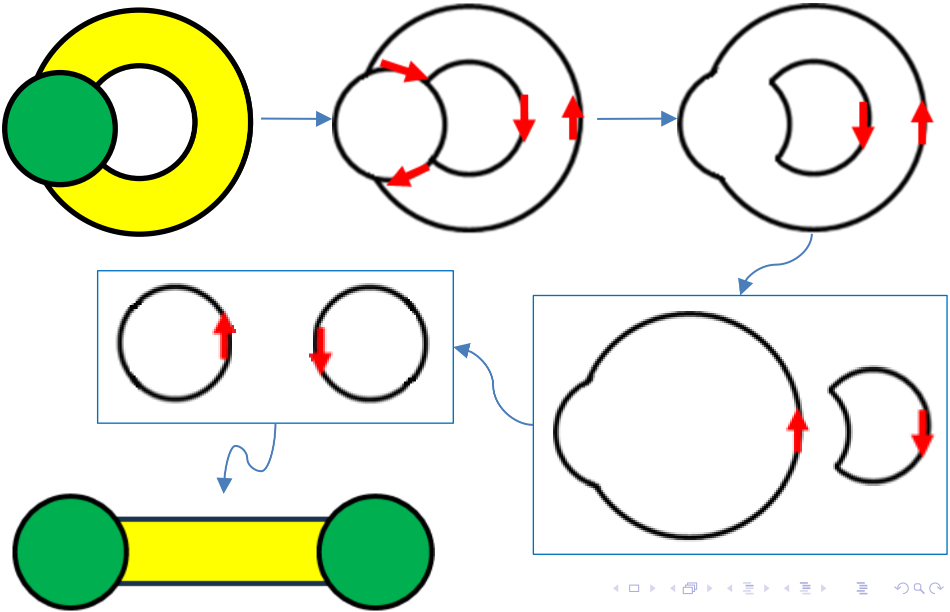
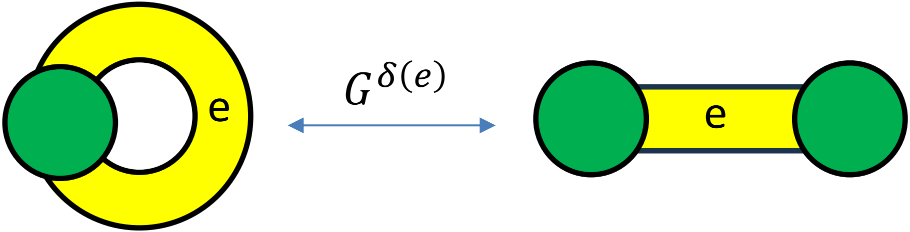
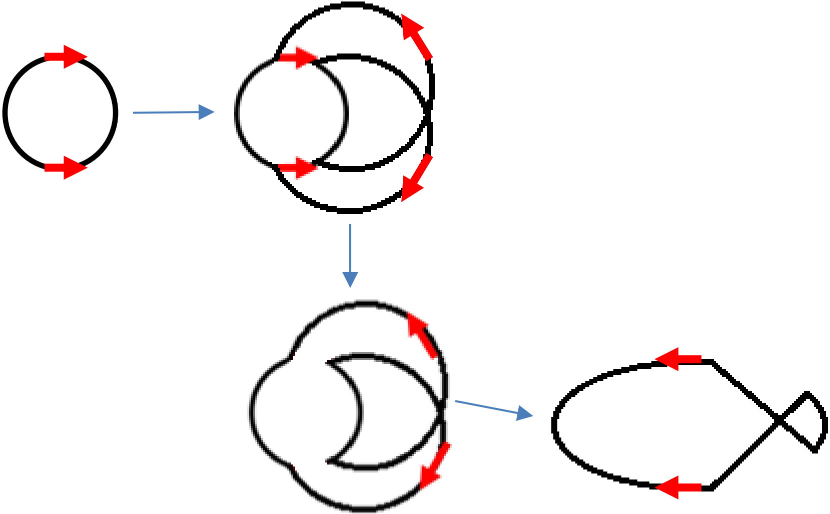


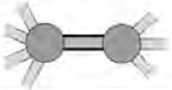
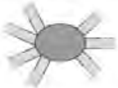
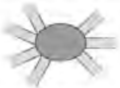
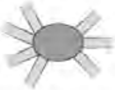
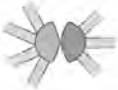
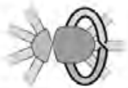
The table of Partial Dual



The table of Partial Dual



The table of Partial Dual

	Non-loop	Non-orientable loop	Orientable loop
G			
$G\backslash e$			
$G/e = G^{\delta(e)}\backslash e$			
$G^{\delta(e)}$			

Chun, C. and Moffatt, I. and Noble, Steven and Rueckriemen, R. (2018) On the interplay between embedded graphs and delta-matroids. Proceedings of the London Mathematical Society 118 (3), pp. 675-700. ISSN 0024-6115.

Some Lemmas

(1) Suppose that an edge e does not belong to A, then $G^{\delta(A\cup\{e\})} = \left(G^{\delta(A)}\right)^{\delta(e)}$

(2) $G = \left(G^{\delta(A)}\right)^{\delta(A)}$

(3) $\left(G^{\delta(A)}\right)^{\delta(B)} = G^{\delta(A\triangle B)}$

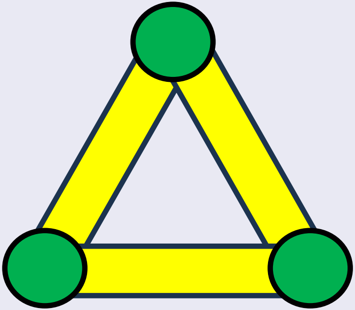
(4) Suppose that an edge e does not belong to A, then $(G/e)^{\delta(A)} = G^{\delta(A\cup\{e\})}\backslash e$

(5) Suppose that an edge e does not belong to A, then $(G\backslash e)^{\delta(A)} = G^{\delta(A\cup\{e\})}/e = G^{\delta(A)}\backslash e$

Spanning subgraph

Definition:
Let $G = (V, E)$ be a ribbon graph. G' is a Spanning subgraph of G iff $G' = (V, E')$ with some $E' \subseteq E$.

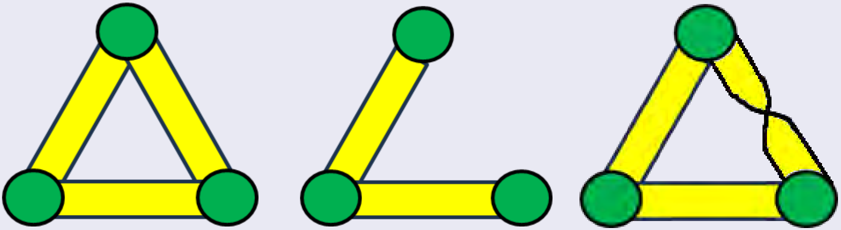
Examples:



Spanning Quasi-tree

Definition:
Let $G = (V, E)$ be a ribbon graph, and let G' be a Spanning subgraph of G. If G' has exactly one boundary component, then G' is a *spanning quasi-tree* of G

Examples:



Delta-Matroid

A set system:
$D = (E, \mathcal{F})$, where $\forall F \in \mathcal{F}, F \subseteq E$

ground set

feasible set

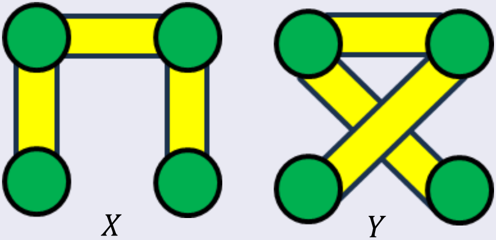
Definition:
A *delta-matroid* is a proper set system $D = (E, \mathcal{F})$ that satisfies the Symmetric Exchange Axiom

Symmetric Exchange Axiom:
For $X, Y \in \mathcal{F}$; for $\forall u \in X \triangle Y, \exists v \in X \triangle Y, \text{s.t.}\, X \triangle \{u, v\} \in \mathcal{F}$

Delta-Matroid

Definition:
Let $G = (V, E)$ be a ribbon graph. And let
$\mathcal{F} := \{F \subseteq E | F$ is the edge set of a spanning quasi−tree of $G\}$
We call $D(G) = (E, \mathcal{F})$ the *delta-matroid* of G.

Verification:



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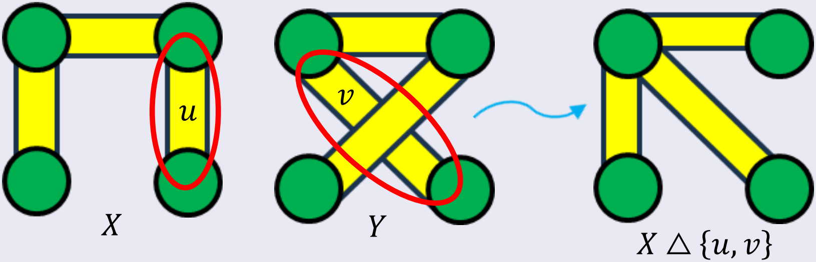


Twist of Delta-Matroids

Definition:
For set system $D = (E, \mathcal{F})$,
$r_{min}(D) = \min\{|F| : F \in \mathcal{F}\}$; $r_{max}(D) = \max\{|F| : F \in \mathcal{F}\}$.
The width of D is defined as $w(D) = r_{max}(D) - r_{min}(D)$.

Definition:
For set system $D = (E, \mathcal{F})$, $D_{min} = (E, \mathcal{F}_{min})$ with
$\mathcal{F}_{min} := \{F \in \mathcal{F} : |F| = r_{min}\}$. D_{max} is defined similarly.

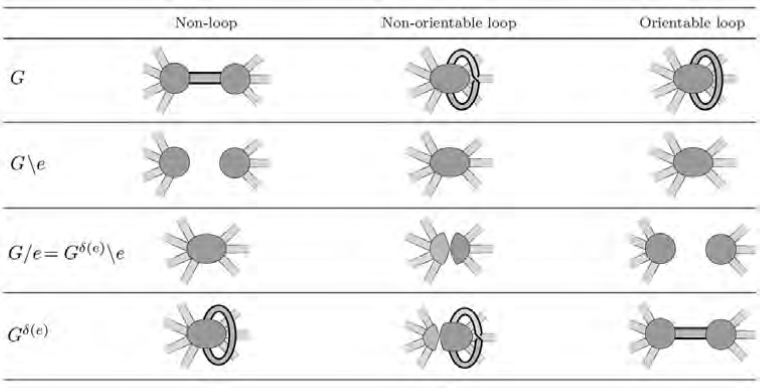
Definition:
Let $D = (E, \mathcal{F})$ be a set system. For $A \subseteq E$, the twist of D with respect to A, denoted by $D * A$, is given by

$$D * A = (E, \{A \triangle X | X \in \mathcal{F}\})$$

An important proposition

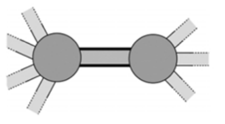
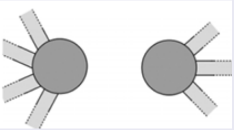
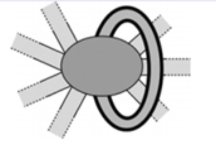
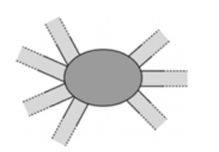
Proposition:
Let $G = (V, E)$ be a ribbon graph,and $A \subseteq E$, then
$$D(G) * A = D\left(G^{\delta(A)}\right)$$

Proof:



An important proposition

Proof (continued):

	$e \in F$	$e \notin F$
G		
$G^{\delta(e)}$		

An important proposition

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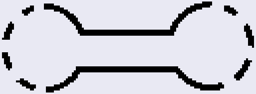
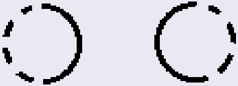
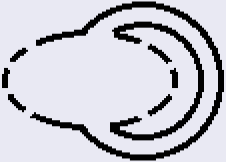
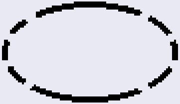
G		
$G^{\delta(e)}$		

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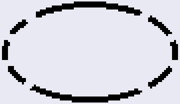
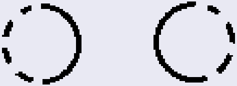
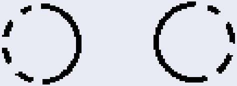
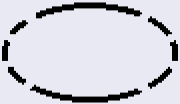
G		
$G^{\delta(e)}$		

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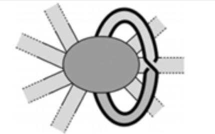
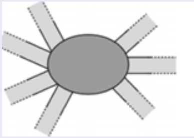
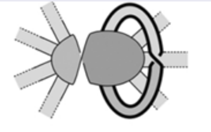
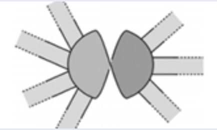
G		
$G^{\delta(e)}$		

An important proposition

Proof (continued):

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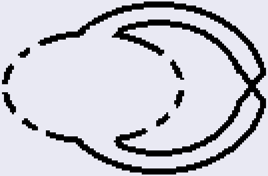
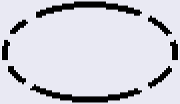
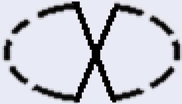
G		
$G^{\delta(e)}$		

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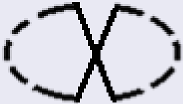
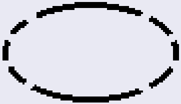
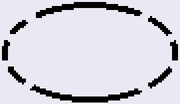
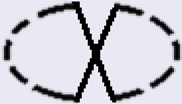
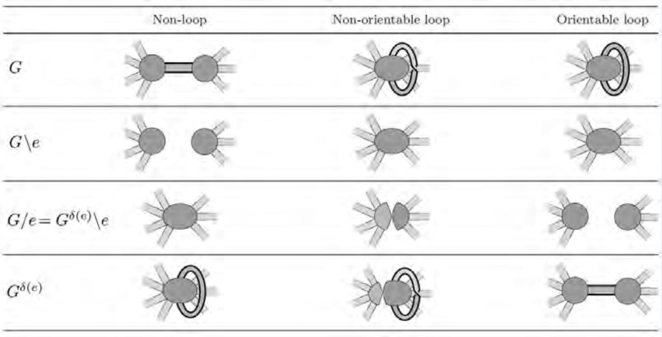
Proof (continued):

	$e \in F$	$e \notin F$
G		
$G^{\delta(e)}$		

An important proposition

Proof (continued):



Recall the previous lemma:

(1) Suppose that an edge e does not belong to A, then

$$G^{\delta(A\cup\{e\})} = \left(G^{\delta(A)}\right)^{\delta(e)}$$

We can finish the proof by induction of any $A \subseteq E$.

Q.E.D

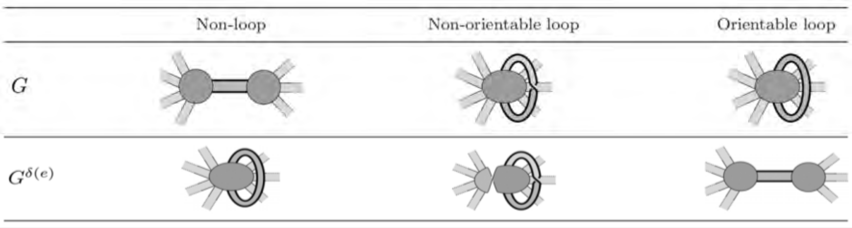
Even more

Lemma.
Let G be a ribbon graph, the Euler genus is invariant under dual, $\gamma(G) = \gamma(G^*)$

Proof:
Recall the Euler formula:
$\gamma(G) = 2k(G) - v(G) + e(G) - f(G)$.
Induction basis: $e(G) = 1$
This proof is easy, we can just the following 3 cases
Case 1: the primal type of e is p (non-loop)
Case 2: the primal type of e is u (untwisted loop)
Case 3: the primal type of e is p (twisted loop)
Induction hypothesis: The statement is true for $|E| = n$, we want to prove the same statement for $E \cup e$ where $e \notin E$.

Even more

Proof (continued):
Induction step:



We can just look at the boundary components, and then prove the statement for all the three cases.

Even more

Lemma.
Let G be a ribbon graph, and Q be a spanning quasi-tree of G. Then
(1) $0 \leq \gamma(Q) \leq \gamma(G)$.
(2) $\gamma(Q) = 0$ if and only if Q is a spanning tree of G.
(3) $\mathrm{Q} = (V(G), A)$ is a spanning quasi-tree of G if and only if $\mathrm{Q}' = (V(G^*), A^c)$ is a spanning quasi-tree of G^*.
And $\gamma(G) = \gamma(Q) + \gamma(Q')$.

Proof:
(1) And (2) are trivial by Euler's formula
$\gamma(G) = 2k(G) - v(G) + e(G) - f(G)$

Even more

Proof (continued):

(3) $G^* = G^{\delta(A\cup A^c)} = G^{\delta(A\triangle A^c)} = \left(G^{\delta(A)}\right)^{\delta(A^c)}$

Note that $G^{\delta(A)}$ is a bouquet, it is obvious that A^c has only one vertex in $G^{\delta(A)}$, and thus A^c has only one boundary component in $\left(G^{\delta(A)}\right)^{\delta(A^c)}$. Thus, $(V(G^*), A^c)$ is a spanning quasi-tree of G^*. The converse direction can be get form dual immediately.

Also, since $\gamma(Q) = 1 - v(Q) + e(Q) = 1 - v(G) + |A|$, and $\gamma(Q') = 1 - v(Q') + e(Q') = 1 - v(G^*) + e(G) - |A|$, where $v(G^*) = f(G)$. Thus,
$\gamma(Q) + \gamma(Q') = 2 - v(G) + e(G) - f(G) = \gamma(G)$.

Q.E.D

Even more

Proof (more rigorous):
To rigorously prove (3), we need the lemma:

Lemma:
(1) If A is the edge set of a spanning quasi-tree of G, then $G^{\delta(A)}$ is a bouquet.
(2) If G is a bouquet, then for any $A \subseteq E(G)$, A forms a spanning quasi-tree of $G^{\delta(A)}$.

Proof of the lemma:
(1) Let A be the set of edges of a quasi-tree of G, then $D\left(G^{\delta(A)}\right) = D(G) * A$. By the definition, $A \in \mathcal{F}(D)$.
So, $\emptyset \in \mathcal{F}(D * A)$. Thus, $v\left(G^{\delta(A)}\right) = 1$, which means bouquet.
(2) Let G be a bouquet, then $\emptyset \in \mathcal{F}(D)$. And thus $A \in \mathcal{F}(D * A)$.
Hence, A forms a quasi-tree of $G^{\delta(A)}$. Q.E.D

Theorem.
The width of a graphic delta-matroid is equal to the Euler genus of the underlying ribbon graph, that is, $\gamma(G) = w(D)$.

Proof:
By the previous lemma, $\gamma(G) = \gamma(Q_{max}) + \gamma(Q'_{min})$.
Also, since Q_{min} and Q'_{min} are spanning tree, we have $\gamma(Q'_{min}) = \gamma(Q_{min}) = 0$. Thus, we have $\gamma(G) = \gamma(Q_{max})$.
By Euler's formula, we have $\gamma(Q_{max}) = |F_{max}| - v(G) + 1$.
By the definition of spanning-tree, $|F_{min}| = v(G) - 1$.
Thus, $\gamma(G) = \gamma(Q_{max}) = |F_{max}| - |F_{min}| = r_{max} - r_{min} = w$.
Q.E.D

Referrence

[1] Chun, C., Moffatt, I., Noble, S. D., & Rueckriemen, R. (2019).
Matroids, delta-matroids and embedded graphs.
Journal of Combinatorial Theory, Series A, 167, 7–59.
https://doi.org/10.1016/j.jcta.2019.02.023